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Perturbative Oscillation Theorems for Jacobi equations

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"The progress of mathematics can be viewed as progress from the infinite to the **finite**.*"*

Gian-Carlo Rota

Contents

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CHAPTER 1

Introduction

The goal of the present thesis is to determine whether the number of eigenvalues below the essential spectrum of the Jacobi operators on $\ell^2(\mathbb{N})$ associated with

$$(\tau u)(n) = a(n)u(n+1) + a(n-1)u(n-1) - b(n)u(n), \tag{1.1}$$

where

$$a(n) \in \mathbb{R}\backslash\{0\}, \quad b(n) \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{1.2}$$

is finite or not.

We will assume $a(n) < 0$ (which is no restriction by [**20**], Lemma 1.6). One of the main cases of interest is if $a(n) = -1$ and one usually starts with the operator H_0 associated with $b_0 = 2$. The spectrum is given by $\sigma(H_0) = [0, 4]$. In particular, there are no eigenvalues below the essential spectrum. Perturbing b_0 we can add any finite number of eigenvalues even if our perturbation is of compact support. However, the question is, can we at least determine whether the number of eigenvalues is finite or not, by looking at the asymptotics of the perturbation? Moreover, what is the precise asymptotics separating the two cases?

The natural tool for investigating such questions is oscillation theory since finiteness of the number of eigenvalues is equivalent to the operator being nonoscillatory. This fact appears first in [**6**]. The precise relation between the number of eigenvalues and the number of *nodes* was established only recently by one of us in [**19**].

In the case of Sturm-Liouville (SL) operators there is a famous theorem by Kneser [**13**] which gives a simple and beautiful answer to this question, with many subsequent extensions by others. The most recent one being by [**5**], who give a unified result containing all previously known ones as special cases. However, much to our surprise, in case of Jacobi operators not even Kneser's result which is more than one hundred years old is known! Our present paper aims at filling this gap.

But first let us review the proof of Kneser's theorem and explain why the discrete case cannot be handled analogously. In the SL case the key idea is that the SL equation

$$\tau_0 u = -\frac{d^2 u}{dx^2} + \frac{\mu u}{x^2} = 0 \tag{1.3}$$

is of Euler type and hence explicitly solvable with a fundamental system given by

$$x^{\frac{1}{2} \pm \sqrt{\mu + \frac{1}{4}}}. \tag{1.4}$$

Hence there are two cases to distinguish. If $\mu \geq -1/4$ all solutions are nonoscillatory. If $\mu < -1/4$ one has to take real/imaginary parts and all solutions are oscillatory. Hence a straightforward application of Sturm's comparison theorem between τ_0 and

$$\tau u = -\frac{d^2 u}{dx^2} + q(x)u(x) \tag{1.5}$$

yields

$$\lim_{x\to\infty} \begin{matrix} \inf \\ \sup \end{matrix} \big(x^2 q(x)\big) \begin{matrix} > \\ < \end{matrix} -\frac{1}{4} \text{ implies } \begin{matrix} \text{nonoscillation} \\ \text{oscillation} \end{matrix} \text{ of } \tau \text{ near } \infty. \tag{1.6}$$

Since Sturm's comparison theorem is also available for Jacobi operators (see, e.g, [**20**], Lemma 4.4) it seems easy to generalize this result by considering the discrete Euler equation

$$u(n+1) - 2u(n) + u(n-1) - \frac{\mu}{n(n-1)} u(n-1) = 0. \tag{1.7}$$

However, unfortunately, this equation is not symmetric! The corresponding results for this equation can be found as special cases in [**1**], Section 6.11. Hence a straightforward generalization is not possible.

Several results in this direction have been obtained by Hinton and Lewis [**9**] and Hooker and Patula [**10**] (see also [**8**], [**11**], [**16**], and [**17**]). However, the generalization of Kneser's results remained unknown.

Our present paper was motivated by the work of Gesztesy and Ünal mentioned earlier. In fact, it can be viewed as a discrete generalization of their results. However, again a straightforward generalization is not possible since their proofs also rely on explicit solubility of the involved equations.

CHAPTER 2

Jacobi operators: Why?

The aim of this chapter is to give a brief introduction into Jacobi equations and Jacobi operators and to set the stage. Jacobi operators appear in a variety of applications. They can be viewed as the discrete analogue of Sturm-Liouville operators and their investigation has many similarities with Sturm-Liouville theory. Spectral and inverse spectral theory plays a fundamental role in the investigation of completely integrable nonlinear lattices, in particular the Toda lattice. In the next two sections we present the relations between Jacobi operators and Sturm-Liouville operators and the Toda lattice.

2.1. Sturm-Liouville and Jacobi equations

In the following sense Jacobi equations are the discrete analogue of Sturm-Liouville equations. An equation of the form

$$(p(x)u'(x))' + q(x)u(x) = 0, \tag{2.1}$$

where we assume $p(x) > 0$ in $[a, b]$ and $p(x), q(x)$ are continuous on $[a, b]$, is a so called **Sturm-Liouville equation**. For small $h = \frac{b-a}{n}$ we can approximate the derivation of $u(x)$ by

$$u'(x) \approx \frac{u(x) - u(x-h)}{h}. \tag{2.2}$$

This yields the following for a Sturm-Liouville equation,

$$\begin{aligned}(p(x)u'(x))' &\approx p(x+h)\frac{u(x+h)-u(x)}{h} - p(x)\frac{u(x)-u(x-h)}{h}\\ &= \frac{1}{h^2}\left[p(x+h)u(x+h) - (p(x+h)+p(x))u(x) + p(x)u(x-h)\right].\end{aligned}$$

In our discrete model of a Sturm-Liouville equation we investigate its behavior at the points $\{a, a+h, a+2h, \cdots, a+nh = b\}$ in the interval $[a, b]$ and therefore our solution $u(x)$ will be considered on the set $\{a, a+h, a+2h, \cdots, a+nh = b\}$, i.e. we set

$$y(t) = u(a+th), \tag{2.3}$$

$$p(t-1) = p(a+th), \tag{2.4}$$

$$q(t) = h^2 q(a+th), \tag{2.5}$$

for $t \in [0, n]$. Now the discrete Sturm-Liouville equations reads as

$$p(t)y(t+1) - (p(t)+p(t-1))y(t) + p(t-1)y(t-1) + q(t)y(t) \approx 0. \tag{2.6}$$

Finally, we can write this in the form

$$\Delta(p(t-1)\Delta y(t-1)) + q(t)y(t) \approx 0. \tag{2.7}$$

The last two equations are so called **Jacobi equations**. The last equation establishes a closer connection between Sturm-Liouville and Jacobi equations, since if one takes the forward difference operator $\Delta u(n) = u(n+1) - u(n)$ as the discrete analogue of the derivation operator. Many theorems about differential equations can be discretized by substituting Δ for the derivation operator but not all relations between differential operators allow such a straightforward translation to difference operators.

2.2. The Toda lattice

This section has no relation to our further investigations. We want to summarize some material about nonlinear equations and Toda lattices to obtain another approach to Jacobi equations and to justify the study of Jacobi equations with a highly nontrivial application. The presentation relies on Chapter 12 of [**20**]. In 1955 E. Fermi, J. Pasta and S. Ulam did one of the first experiments in which the computer played a central role. They considered a one-dimensional dynamical system of 64 particles with a nonlinear nearest neighbor interaction. They excited the lowest mode and looked at the behavior of the system. It was shown that some energy is exchanged from the lowest mode to some higher modes, but periodically almost all energy flows back to the lowest mode. The difference between the initial condition and the condition after one period can be described by the quasiperiodic flow on a torus-like-surface, [**2**]. Ten years later N.J. Zabusky and M.D. Kruskal showed that the behavior observed by FPU can be described by **solitons**. All this research led people to consider nonlinear lattices more thoroughly. The equations of a nonlinear lattice are readily derived, but it proved to be hard to find a nonlinear lattice that is completely integrable. Trial and error led M. Toda to consider a lattice with an exponential spring potential [**21**]. The solutions are explicitly described by elliptic functions that seem to be the natural extension of harmonic functions in a linear lattice. One finds solitons and cnoidal waves as a special class of solutions. The Toda lattice can be analyzed by spectral methods [**20**] and this generates a connection between the Toda flow and the symmetric eigenvalue problem.

The **Toda lattice** is a simple model for a nonlinear one-dimensional crystal. It describes the motion of a chain of particles with nearest neighbor interaction. The equation of motion of such a system is given by

$$m\frac{d^2}{dt^2}x(n,t) = V'(x(n+1,t) - x(n,t)) - V'(x(n,t) - x(n-1,t)) \tag{2.8}$$

where m denotes the mass of each particle and $x(n,t)$ is the displacement of the $n-th$ particle from its equilibrium position and $V(r)$ is the interaction potential. As discovered by M. Toda, this system gets particularly interesting if one chooses an exponential interaction,

$$V(r) = \frac{m\rho^2}{\tau^2}\big(e^{-r/\rho} + \frac{r}{\rho} - 1\big) = \frac{m\rho^2}{\tau^2}\big((\frac{r}{\rho})^2 + O(\frac{r}{\rho})^3\big),\ \tau,\rho \in \mathbb{R}. \tag{2.9}$$

This model is of course only valid as long as relative displacement is not too large, i.e. at least smaller than the distance of the particles in the equilibrium position. For small displacements it is equal to a harmonic crystal with force constant $\frac{m}{\tau^2}$.

REMARK 2.1. *If τ and ρ are positive, then the repulsive force is much stronger than the attractive force. We can consider the limit where the repulsive force becomes infinitely sharp and the attractive force is not present. This is obtained by letting $b \to \infty$ and $a \to 0$ but ab finite. It can be shown that the particles move freely until there is a collision, during which velocity is exchanged. This is the so called* **hard sphere limit**. *Due to the asymmetric form of the potential in the Toda equation, it can be shown that if energy is put in the chain, then the average distance between neighbors will increase if $\rho > 0$ and decrease if $\rho < 0$. This behavior is not present in linear chains, where the average distance between particles is fixed to the equilibrium distance.*

After a scaling transformation $t \mapsto t/\tau, x \mapsto x/\rho$, we can assume $m = \tau = \rho = 1$. If we suppose $x(n,t) - x(n-1,t) \to 0, \dot{x}(n,t) \to 0$ sufficiently fast as $n \to \infty$, we can introduce the Hamiltonian ($q = x, p = \dot{x}$)

$$H(p,q) = \sum_{n\in\mathbb{Z}} \big(\frac{p(n,t)^2}{2} + e^{-(q(n+1,t)-q(n,t))} - 1\big) \tag{2.10}$$

and rewrite the equation of motion in the Hamiltonian form

$$\begin{aligned} \frac{d}{dt}p(n,t) &= -\frac{\partial H(p,q)}{\partial q(n,t)} && (2.11) \\ &= e^{-(q(n,t)-q(n-1,t))} - e^{-(q(n+1,t)-q(n,t))}, && (2.12) \end{aligned}$$

$$\frac{d}{dt}q(n,t) = \frac{\partial H(p,q)}{\partial p(n,t)} = p(n,t). \tag{2.13}$$

We remark that these equations are invariant under the transformation

$$p(n,t) \mapsto p(n,t) + p_0, \quad q(n,t) \mapsto q(n,t) + q_0 + p_0 t, \ (p_0, q_0) \in \mathbb{R}^2, \tag{2.14}$$

which reflects the fact that the dynamics remains unchanged by a uniform motion or the entire crystal. The fact which makes the Toda lattice particularly interesting is the existence of soliton solutions. These are pulslike waves traveling through the crystal without changing their shape. Such solitons are rather special since from a generic linear equation one would expect spreading of wave packets and from a generic nonlinear equation one would expect that solutions only exist for a finite time (breaking of waves). The simplest example of such a solitary wave is a one-soliton solution

$$q_1(n,t) = q_0 - \log \frac{1 + \gamma \exp(-2\kappa n \pm 2\sinh(\kappa)t)}{1 + \gamma \exp(-2\kappa(n-1) + 2\sinh(\kappa)t)}, \ \kappa, \gamma > 0. \tag{2.15}$$

It describes a single bump traveling through the crystal with speed $\pm\sinh(\kappa)/\kappa$ and width proportional to $\frac{1}{\kappa}$. That is, the smaller the soliton the faster it propagates. It results in a total displacement of the crystal, which can be equivalently interpreted as the total compression of the crystal around the bump. The total moment and energy are given by

$$\sum_{n\in\mathbb{Z}} p_1(n,t) = 2\sinh(\kappa), \quad H(p_1, q_1) = 2\sinh(\kappa)\cosh(\kappa) - \kappa). \tag{2.16}$$

Existence of such solutions is usually connected to complete integrability of the system which is indeed the case here. To see this, we introduce Flaschka's variables

$$a(n,t) = \frac{1}{2} e^{-(q(n+1,t)-q(n,t))/2}, \quad b(n,t) = -\frac{1}{2} p(n,t) \tag{2.17}$$

and obtain the form most convenient for us

$$\dot{a}(n,t) = a(n,t)\big(b(n+1,t) - b(n,t)\big) \tag{2.18}$$

$$\dot{b}(n,t) = 2\big(a(n,t)^2 - a(n-1,t)^2\big). \tag{2.19}$$

To show complete integrability it suffices to find a so-called Lax pair, that is, two operators $H(t), P(t)$ such that the Lax equation

$$\frac{d}{dt} H(t) = P(t)H(t) - H(t)P(t) \tag{2.20}$$

is equivalent to the equation of motion in Flaschka variables. One can easily convince oneself that the choice

$$H(t): \quad \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \tag{2.21}$$

$$f(n) \mapsto a(n,t)f(n+1,t) + a(n-1,t)f(n-1,t) + b(n,t)f(n) \tag{2.22}$$

$$P(t): \quad \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \tag{2.23}$$

$$a(n,t)f(n+1) - a(n-1,t)f(n-1) \tag{2.24}$$

does the trick. Now the Lax equation implies that the operators $H(t)$ for different $t \in \mathbb{R}$ are all unitarily equivalent and that

$$tr(H(t)^j - H_0^j), \ j \in \mathbb{N}, \tag{2.25}$$

are conserved quantities, where H_0 is the operator corresponding to the constant solution $a_0(n,t) = \frac{1}{2}, b_0(n,t) = 0$. The operator $H(t)$ is a **Jacobi operator**. For more details on the Toda lattice and spectral theory of Jacobi operators we refer the reader to the monograph [**20**].

CHAPTER 3

Jacobi operators

3.1. General properties of Jacobi operators

In the sequel we set the notation and state some elementary properties of Jacobi operators. We follow [**20**], Chapter 1. We consider the vector space of real-valued sequences V_s and some of its subspaces.

DEFINITION 3.1. (i) $\ell^1(\mathbb{N}) = \{u \in V_s | \|u\|_1 = \sum_{n=0}^{\infty} |u(n)| < \infty\}$
(ii) $\ell^2(\mathbb{N}) = \{u \in V_s | \|u\|_2 = \sum_{n=0}^{\infty} |u(n)|^2 < \infty\}$
(iii) $\ell^\infty(\mathbb{N}) = \{u \in V_s | \|u\|_\infty = \sup_{n\in\mathbb{N}} u(n) < \infty\}$
(iv) $\ell_0(\mathbb{N}) = \{u \in V_s | \lim_{n\to\infty} u(n) = 0\}$

And the following difference operators acting on $u \in V_s$:
(i) forward difference: $(\Delta u)(n) = u(n+1) - u(n)$,
(ii) backward difference: $(\nabla u)(n) = u(n) - u(n-1)$.

REMARK 3.2. *The vector spaces (i)-(iv) are Banach spaces. The space (ii) is actually a Hilbert space with respect to the scalar product*

$$\langle u, v\rangle = \sum_{n=0}^{\infty} u(n)v(n). \tag{3.1}$$

As we mentioned in the preface and the introduction we are interested in *second order difference expressions*

$$a(n+1)u(n+1) + b(n)u(n) + c(n-1)u(n-1), \tag{3.2}$$

or equivalently

$$\nabla(p(n)\Delta u)(n) + q(n)u(n), \tag{3.3}$$

for suitable sequences $p(n), q(n)$. Most of the time we use the operator free expression (3.2), but at some places it is more convenient to use the operator form. Appendix 1 contains some connections between difference operators and difference expressions.

A difference expression $a(n-1)u(n-1) + b(n)u(n) + c(n+1)u(n+1)$ can be associated with the following matrix X and vector f

$$X = \begin{pmatrix} a(0) & b(1) & c(2) & & \\ & a(1) & b(2) & c(3) & \\ & \ddots & \ddots & \ddots & \end{pmatrix}. \tag{3.4}$$

$$u = \begin{pmatrix} u(0) \\ u(1) \\ \vdots \end{pmatrix}, \tag{3.5}$$

which yields for our difference expression

$$Xf. \tag{3.6}$$

X is a tridiagonal matrix, but which choice of the sequence $c(n)$ yields a symmetric matrix? The answer follows in an apparent manner from the condition $X = X^T$, which is equivalent to $c(n+1) = a(n)$, i.e.

$$a(n)u(n+1) + a(n-1)u(n-1) + b(n)u(n). \tag{3.7}$$

bywith H. .

Definition 3.3.

(i) An expression of the form $\tau u(n) = a(n)u(n+1) + a(n-1)u(n-1) - b(n)u(n)$ for $n \in \mathbb{N}$ is called a **Jacobi difference expression** associated to the sequences $a, b \in \ell(\mathbb{N})$.
(ii) We call $\tau u(n) = 0$ a **Jacobi equation** on $\mathbb{N}$, with $z \in \mathbb{C}$ and $n \in \mathbb{N}$.
(iii) The matrix H associated to a Jacobi difference equation is called a **semi-infinite Jacobi matrix**.

Jacobi equations are second order linear difference equations, thus they have two linearly independent solutions and solutions are determined by two initial conditions $u(n_0)$ and $u(n_0+1)$. A short calculation shows the validity of **Green's formula**

$$\sum_{j=m}^{n} \big(u(j)(\tau v)(j) - v(j(\tau u)(j)\big) = W_n(u,v) - W_m(u,v) \tag{3.8}$$

for $u, v \in \ell(\mathbb{N})$, where we have introduced the **Wronskian**

$$W_n(u,v) = a(n)\big(u(n)v(n+1) - u(n+1)v(n)\big). \tag{3.9}$$

The Wronskian is much more than a suitable abbreviation as the following theorem shows.

Theorem 3.4. *Let u, v be solutions of $\tau u(n) = 0$, then the following conditions are equivalent:*

(i) *u, v are linearly independent,*
(ii) *$W_n(u,v) = 0$ for some n,*
(iii) *$W_n(u,v) = 0$ for all n.*

The Wronskian also indicates linear independence of solutions of Jacobi equations.

Theorem 3.5. *Let u, v be solutions of $\tau u(n) = 0$ then the following are equivalent:*

(i) *u, v are linearly independent.*
(ii) *$W_n(u,v) \neq 0$.*

Moreover the Wronskian is constant and therefore we have a $C \in \mathbb{R}$ with

$$W(u,v) = C. \tag{3.10}$$

Another example for the usefulness of the Wronskian is **Lagrange's Identity**

$$u(n)(\tau v)(n) - v(n)(\tau u)(n) = \Delta\big(W_{n-1}(u,v)\big), \tag{3.11}$$

which follows from some easy manipulation of Jacobi operators. As an application we state a theorem due to G. Polya.

THEOREM 3.6. *Polya Assume v be a positive solution of $\tau u(n) = 0$, then there exist sequences ρ_1, ρ_2 with $\rho_1, \rho_2 > 0$, such that for any sequence u on $\mathbb{N}$*

$$\tau u(n) = \frac{1}{v(n)} \Delta\big(\rho_2(n) \Delta(\rho_1(n-1) u(n-1))\big). \tag{3.12}$$

PROOF. Since v is a positive solution of $\tau u(n) = 0$, we have by Lagrange's identity that

$$\tau u(n) = \frac{1}{v(n)} \Delta\big(W_{n-1}(u, v)\big). \tag{3.13}$$

We also have

$$\Delta\big(\frac{u(n-1)}{v(n-1)}\big) = \frac{W_{n-1}(u, v)}{a(n-1) v(n-1) v(n)}, \tag{3.14}$$

which implies

$$\tau u(n) = \rho_1(n) \Delta\big(\rho_2(n) \Delta(\rho_1(n-1) u(n-1))\big), \tag{3.15}$$

where

$$\rho_1(n) = \frac{1}{v(n)}, \quad \rho_2(n) = (n-1) v(n-1) v(n). \tag{3.16}$$

□

Now we draw some consequences out from the fact that the space of solutions of $\tau u(n) = 0$ is two dimensional. Therefore we can pick two linearly independent solutions of $\tau u(n) = 0$ and write any solution u as a linear combination of these two solutions

$$u(n) = \frac{W(u, s)}{W(c, s)} c(n) - \frac{W(u, c)}{W(c, s)} s(n). \tag{3.17}$$

For this purpose it is convenient to introduce the following **fundamental solutions** $c, s \in \ell(\mathbb{N})$

$$\tau c(n, n_0) = 0, \quad \tau s(n, n_0) = 0, \tag{3.18}$$

fulfilling the **initial conditions**

$$s(n, n_0) = 0, \quad s(n, n_0 + 1) = 1, \tag{3.19}$$

$$c(n, n_0) = 1, \quad c(n, n_0 + 1) = 0. \tag{3.20}$$

The equation

$$\tau u(n) = v(n) \tag{3.21}$$

for $v \in \ell(\mathbb{N})$ is referred to as **Inhomogeneous Jacobi equation**. Its solution can be completely reduced to the solution of the corresponding **homogeneous Jacobi equation**

$$\tau u(n) = 0. \tag{3.22}$$

We introduce

$$K(n, m) = \frac{s(n, m)}{a(m)}, \tag{3.23}$$

where $s(n, n_0)$ is the fundamental solution of

$$\tau s(n, n_0) = 0. \tag{3.24}$$

Then the sequence

$$u(n) = u(n_0) c(n, n_0) + u(n_0 + 1) s(n, n_0) + \sum_{m=n_0+1}^{n-1} K(n, m) v(m), \tag{3.25}$$

where we assume $n > n_0$ is a solution of the inhomogeneous Jacobi equation. The summation kernel has the following properties

(i) $K(n,n) = 0$,
(ii) $K(n+1,n) = a(n)^{-1}$,
(iii) $K(n,m) = K(m,n)$,
(iv) $K(n,m) = \frac{u(m)v(n)-u(n)v(m)}{W(u,v)}$ for any pair $u(n), v(n)$ of linearly independent solutions of $\tau u(n) = 0$.

For $v(n) = \hat{u}(n)$ the summation kernel simplifies to

$$K(n,m) = u(n)u(m) \sum_{j=m+1}^{n-1} \frac{1}{a(j)u(j)u(j+1)} \tag{3.26}$$

and

$$u(n) = u(n_0)c(n,n_0) + u(n_0+1)s(n,n_0) + \sum_{m=n_0+1}^{n-1} K(n,m)\hat{u}(m) \tag{3.27}$$

is a solution of

$$\tau u(n) = \hat{u}(n). \tag{3.28}$$

In the proof of our main theorem a similar inhomogeneous Jacobi equation will occur. Another ingredient in the proof of our main theorem is the following fact about Jacobi equations.

3.2. Variation of constants

Let $u_1(n) \neq 0$ be a solution of $\tau u(n) = 0$ then one wants to construct another linearly independent solution $u_2(n)$. One way is the following:

$$\begin{aligned}
\Delta\Big(\frac{u_2(n)}{u_1(n)}\Big) &= \frac{u_2(n+1)u_1(n) - u_2(n)u_1(n+1)}{u_1(n)u_1(n+1)} \\
&= \frac{W_n(u_1,u_2)}{a(n)u(n)u(n+1)} \\
&= \frac{C}{a(n)u_1(n)u_1(n+1)} \\
u_2(n) &= u_1(n)\sum_{j=0}^{n-1} \frac{C}{a(j)u(j)u(j+1)}.
\end{aligned} \tag{3.29}$$

The solution u_2 depends on the structure of $u_1(n)$ and to make this more transparent we denote solutions of the form u_2 with $C = 1$ by $\hat{u}_1(n)$.

Definition 3.7. *Let $u_1(n)$ be a positive solution of $\tau u(n) = 0$, then we set*

$$\hat{u}_1(n) = u_1(n)\sum_{j=0}^{n-1} \frac{1}{a(j)u_1(j)u_1(j+1)} \tag{3.30}$$

for the second linearly independent solution.

REMARK 3.8. *To illustrate the method, we compute $\hat{u}(n)$ for $u(n+1)+u(n-1)-2u(n)=0$. The function $u_1=1$ is a solution then*

$$\hat{u}_1(n) = u_1(n)\sum_{j=0}^{n-1}\frac{1}{u(j)u(j+1)} = 1\cdot(1+...+1) = n. \tag{3.31}$$

3.3. Sum equations

Next we study sum equations especially **Volterra sum equations** because we need them in the proof of our main theorem.

DEFINITION 3.9. *A* **Volterra sum-equation** *is an equation of the form*

$$u(n) = v(n) + \sum_{m=0}^{\infty} K(n,m)h(m)u(m). \tag{3.32}$$

The following theorem is fundamental in the theory of sum-equations and will play a major role in the proof of our oscillation criteria.

THEOREM 3.10. *Suppose there is a sequence $\hat{K}(n,m)$ such that*

$$|K(n,m)| \le \hat{K}(n,m), \qquad \hat{K}(n+1,m) \le \hat{K}(n,m), \qquad \hat{K}(n,.) \in \ell^1(\mathbb{N}) \tag{3.33}$$

and $v \in \ell^1(0,\infty)$ and $w(n) \in \ell^\infty$, then

$$u(n) = v(n) + \sum_{m=0}^{\infty} K(n,m)w(m)u(m) \tag{3.34}$$

has a unique solution u in $\ell^\infty(0,\infty)$. If $v(n)$ and $K(n,m)$ depend continuously on a parameter and if $\hat{K}(n,m)$ does not depend on this parameter, then the solution depends continuously on this parameter.

PROOF. See [**20**], p.126. □

3.4. Spectral theory

In the sequel we assume that a,b are bounded sequences.
Hypothesis H.2.1. Suppose $a,b \in \ell^\infty(\mathbb{N},\mathbb{R})$, $\quad a(n) \neq 0$. Associated with a,b is the **Jacobi operator**

$$\begin{aligned} H : \ell^2(\mathbb{N}) &\to \ell^2(\mathbb{N}) \\ u &\mapsto \tau u. \end{aligned}$$

DEFINITION 3.11. *$\|H\|$ denotes the operator norm of H.*

THEOREM 3.12. (i) $\|a\|_\infty \le \|H\|$
(ii) $\|b\|_\infty \le \|H\|$
(iii) $\|H\| \le 2\|a\|_\infty + \|b\|_\infty$.
(iv) *H is self-adjoint.*

PROOF. (i) and (ii) follow from the definition. (iii) $a(n-1)^2+a(n)^2+b(n)^2 = \|H\delta_n\| \le \|H\|^2$ and

$$|\langle u, Hu\rangle| \le (2\|a\|_\infty + \|b\|_\infty)\|u\|^2. \tag{3.35}$$

□

REMARK 3.13. *H is bounded if and only if the sequences a, b are bounded.*

H is a bounded self-adjoint operator on a Hilbert space, but such operators are well studied objects and simplify the functional analytical treatment tremendous. The following theorems are a few examples for the well-behavdness of bounded self-adjoint operators.

THEOREM 3.14. *All the eigenvalues of H are real and two eigenvectors of H corresponding to distinct eigenvalues are orthogonal.*

PROOF. If $Hu = zu$ and $f \neq 0$, then

$$z\langle u, u\rangle = \langle Hu, u\rangle = \langle u, Hu\rangle = \langle u, zu\rangle = \overline{z}\langle u, u\rangle \tag{3.36}$$

and hence $z \in \mathbb{R}$. Moreover, if

$$Hu_1 = z_1 u_1, \quad Hu_2 = z_2 u_2, \quad z_1 \neq z_2, \tag{3.37}$$

then

$$z_1\langle u_1, u_2\rangle = \langle Hu_1, u_2\rangle = \langle u_1, Hu_2\rangle = \langle u_1, z_2 u_2\rangle = \overline{z_2}\langle u_1, u_2\rangle \tag{3.38}$$

and hence u_1 and u_2 are orthogonal. □

If one deals with operators, one always wants to know the spectrum, i.e the set

$$\sigma(H) = \{z \in \mathbb{C} |\ (H - z)^{-1}\text{does not exist or is unbounded}\}. \tag{3.39}$$

The spectrum of H is in general a non empty compact subset of $\mathbb{C}$, and consists of

(i) z is an eigenvalue of H, i.e. $Hu = zu$ and $z \in \ell^2(\mathbb{N})$
(ii) z is a point of the continuous spectrum, i.e $(H - z)^{-1}$ exists but fails to be continuous.

For bounded operators H one knows, that the spectrum is contained in the disk of radius $\|H\|$, but in our case we know much more.

LEMMA 3.15. *Let*

$$c_{\pm}(n) = b(n) \pm (|a(n)| + |a(n-1)|) \tag{3.40}$$

Then we have

$$\sigma(H) \subseteq [\inf_{n\in\mathbb{N}} c_-(n), \sup_{n\in\mathbb{N}} c_+(n)] \tag{3.41}$$

PROOF. We will first show that H is bounded from above by $\sup c_+$.

$$\langle u, Hu\rangle = \sum_{n\in\mathbb{N}} \big(-a(n)|u(n+1) - u(n)|^2 + (a(n-1) + a(n) + b(n))|u(n)|^2\big). \tag{3.42}$$

Then $a(n) > 0$ shows

$$\langle u, Hu\rangle \leq \sup_{\in\mathbb{N}} c_+(n)\|u\|^2. \tag{3.43}$$

Similarly, choosing $a(n) < 0$ shows that H is semibounded from below by $\inf c_-$, which completes the proof. □

REMARK 3.16. *For more information about spectral theory of Jacobi equation I refer the reader to* [**20**].

CHAPTER 4

Oscillation Theory

We are interested in the behavior of solutions of $\tau u = 0$. For example the solutions of $u(n+1) + u(n-1) - 2u(n) = 0$ are positive and nonvanishing for $n \in \mathbb{N}$. On the other hand

$$u(n+1) - u(n-1) = 0, \quad n \in \mathbb{N} \tag{4.1}$$

has the solutions $u_1(n) = \cos(\frac{n\pi}{2})$ and $u_2(n) = \sin(\frac{n\pi}{2})$, which possess infinitely many zeros. If a Jacobi difference equation $\tau u = 0$ has a nonvanishing solution $u_1(n)$ then every linearly independent solution $u_2(n)$ is nonvanishing, too. Without loss of generality we assume $u_1(n) > 0$, since if $u_1(n)$ is a solution then $-u_1(n)$ is also solution of $\tau u(n) = 0$. We consider

$$\Delta\Big(\frac{u_2(n)}{u_1(n)}\Big) = \frac{c}{a(n)u(n)u(n+1)}, \tag{4.2}$$

which is of fixed sign and therefore $\frac{u_2(n)}{u_1(n)}$ is monotone, which is not possible if $u_2(n)$ possesses an infinite number of zeros.

Definition 4.1. *Let $u_1(n)$ be a solution of $\tau u = 0$.*

(i) *If for any $n \in \mathbb{N}$ there exists an $n_1 \in \mathbb{N}$ such that $u_1(n_1)u_1(n_1+1) < 0$ then τ is called* **oscillatory***.*
(ii) *If $u_1(n)$ is not oscillatory, then τ is called* **nonoscillatory***.*

Remark 4.2. *There are linear second order difference equations with a positive and an oscillatory solution. For example*

$$\Delta^2 u(n) + \frac{8}{3}\Delta u(n) + \frac{4}{3}u(n) = 0 \tag{4.3}$$

has an oscillatory solution $u(n) = (-1)^n$ and a nonoscillatory solution $v(n) = 3^{-n}$.

If we assume $a(n) < 0$ and τ nonoscillatory, then there exists a positive solution $u_1(n)$ of $\tau u = 0$ and therefore $a(n)u(n+1) + a(n-1)u(n-1)$ is negative but this implies $b(n)u(n) < 0$, but $u(n) > 0$ and therefore $b(n) < 0$. This proves the following

Lemma 4.3. *Let $a(n) < 0$ and $u(n)$ be a positive solution of $\tau u(n) = 0$, then*

(i) *τ is nonoscillatory if and only if $b(n) < 0$.*
(ii) *τ is oscillatory if and only if $b(n)$ is positive.*

4.1. Minimal and maximal solutions

Definition 4.4. *Suppose $a(n) < 0$, $\lambda \leq \sigma(H)$ and let $u(\lambda, n)$ be a solution with $u(\lambda, n) \geq 0$ for $n \in \mathbb{N}$. We call a solution $u(\lambda, n)$* **minimal near ∞** *if $\sum_{j=0}^{\infty} \frac{-1}{a(j)u(j)u(j+1)} = \infty$.*

THEOREM 4.5. *Suppose $a(n) < 0$, $\lambda \le \sigma(H)$ and let $u(\lambda, n)$ be a solution with $u(\lambda, n) \ge 0$ for $n \in \mathbb{N}$. Then the following conditions are equivalent*

(i) *$u(\lambda, n)$ is minimal near ∞.*
(ii) *$\frac{u(\lambda,n)}{v(\lambda,n)} \le \frac{u(\lambda,0)}{v(\lambda,0)}$ for any solution $v(\lambda, n) > 0$ and $n \in \mathbb{N}$.*
(iii) *We have $\lim_{n\to\infty} \frac{u(\lambda,n)}{v(\lambda,n)} = 0$ for a positive solution $v(\lambda, n)$ and $n \in \mathbb{N}$.*

PROOF. See [**20**]. □

REMARK 4.6. *The solution $v(n)$ of the last theorem is called* **minimal near ∞** *and satisfies*

$$\sum_{j}^{\infty} \frac{-1}{a(j)v(j)v(j+1)} \in \ell^1(\mathbb{N}). \tag{4.4}$$

Minimal and maximal solutions are unique up to a constant factor.

In our example $-u(n+1) - u(n-1) + 2u(n) = 0$ the solution $u_1(n) \equiv 1$ is minimal near ∞ and $\hat{u}_1(n) = n$ is maximal near ∞. This is a special case of

THEOREM 4.7. (i) *If τ is nonoscillatory and $u_1(n)$ is a minimal solution near ∞, then*

$$\hat{u}_1(n) = u_1(n) \sum_{j=0}^{n-1} \frac{-1}{a(j)u_1(j)u_1(j+1)} \tag{4.5}$$

is maximal near ∞.

(ii) *If τ is nonoscillatory and $u_2(n)$ is maximal near ∞ then*

$$u_1(n) = u_2(n) \sum_{j=n}^{\infty} \frac{-1}{a(j)u_2(j)u_2(j+1)} \tag{4.6}$$

is minimal near ∞.

PROOF. (i) If $u_1(n)$ is minimal near ∞, i.e.

$$\sum_{j}^{\infty} \frac{-1}{a(j)u(j)u(j+1)} = \infty, \tag{4.7}$$

then $\hat{u}_1(n) = u_1 \sum_{j=0}^{n-1} \frac{-1}{a(j)u(j)u(j+1)}$ is a solution of $\tau u = 0$, which yields

$$\lim_{n\to\infty} \frac{\hat{u}_1(n)}{u_1(n)} = \infty \tag{4.8}$$

and thus $\hat{u}_1(n)$ is maximal near ∞.

(ii) Can be proved similarly.

□

REMARK 4.8. *If all solutions of $\tau u(n) = 0$ are bounded, then the maximal solution is in $\ell_0(\mathbb{N})$.*

In more precious form holds

THEOREM 4.9. *If $\lambda < \sigma(H)$ then there exists a minimal and maximal solution $u_1(n)$ and $u_2(n)$ of $\tau u(n) = 0$ such that*

(i) *$u_1(n) > 0$ and $u_1(n+1) > u_1(n)$,*

(ii) $u_2(n) > 0$ *and* $u_2(n+1) < u_2(n)$.

COROLLARY 4.10.

(i) $\lim_{n\to\infty} u_1(n) = \infty$,
(ii) $\lim_{n\to\infty} u_2(n) = 0$.

Some of the material is from [**1**], Chapter 6.

Now we state some sufficient conditions for τ be oscillatory.

THEOREM 4.11. *If* $b(n) \le \min(a(n-1), a(n))$ *for sufficiently large* $n \in \mathbb{N}$ *then* $\tau u = 0$ *is oscillatory.*

PROOF. Let $u_1(n)$ be a positive solution of $\tau u = 0$. We can assume $b(n) < 0$ for all large $n \in \mathbb{N}$. However $\tau u = 0$ implies that

$$u_1(n+1) < \frac{b(n)}{a(n)} u_1(n), \qquad u_1(n-1) < \frac{b(n)}{a(n-1)} u(n) \tag{4.9}$$

holds for all large $n \in \mathbb{N}$. But, $\frac{b(n)}{a(n)} \le 1$ and $\frac{b(n)}{a(n-1)} \le 1$ yields $u(n+1) < u(n)$ and $u(n-1) < u(n)$ for all large $n \in \mathbb{N}$. This contradiction completes the proof. □

COROLLARY 4.12. *If* $b(n) \le a(n)$ *and if* $a(n)$ *is nonincreasing then* τ *is oscillatory.*

PROOF. Straightforward. □

COROLLARY 4.13. *If* $b(n) \le a(n-1)$ *and* $a(n)$ *is nondecreasing then tau is oscillatory.*

PROOF. Straightforward. □

REMARK 4.14. *The notions of maximal and minimal solutions near* ∞ *are introduced in* [**15**].

If one wants a discrete analogue of Sturm's comparison theorem the notion of a zero of a solution is not sufficient and has to be enlarged. For the following definitions and proofs of the theorems see [**20**].

DEFINITION 4.15. *Let us call a point* $n \in \mathbb{N}$ *a* **node** *of a solution of* $\tau u(n) = 0$, $a(n) < 0$, *if one of the following conditions holds*

(i) $u(n) = 0$
(ii) $a(n)u(n)u(n+1) > 0$

DEFINITION 4.16. *Let us call* τ **oscillatory** *if a solution of* $\tau u(n) = 0$ *has an infinite number of nodes.*

One observes that solutions of Jacobi equations split into two cases

(i) All solutions are nonoscillatory.
(ii) All solutions are oscillatory.

The essential tool in the justification of this observation is Sturm's comparison theorem. We only formulate Sturm's separation and comparison theorem.

DEFINITION 4.17. *A node* n_0 *lies between* m *and* n *if*

(i) $m < n_0 < n$
(ii) $n_0 = m$ *and* $u(m) \neq 0$.

Denote by $\#_{m,n}u$ *the number of nodes between* m *and* n.

THEOREM 4.18. *Sturm's Separation Theorem Let* $m < n$ *be nodes of* u_1 *a solution of* $\tau u(n) = \lambda_1 u(n)$, *or zeros of* $W(u_1, u_2)$ *with* u_2 *a solution of* $\tau u(n) = \lambda_2 u(n)$, *with* $\lambda_1 \leq \lambda_2$, *such that* u_1 *has no further nodes in* $[m, n]$.

(i) u_2 *has at least one node in* $[m, n+1]$.
(ii) $\#_{(m,n)}u_2 \geq \#_{(m,n)}u_1 - 1$.

PROOF. By contradiction. See [**20**]. □

THEOREM 4.19. *Sturm's Comparison Theorem*
Let

$$\tau_1 u(n) = a(n)u(n+1) + a(n-1)u(n-1) - b_1(n)u(n) = 0 \tag{4.10}$$

be nonoscillatory and

$$\tau_2(n) = a(n)u(n+1) + a(n-1)u(n-1) - b_2(n)u(n) = 0, \tag{4.11}$$

where $b_2(n) \geq b_1(n)$ *then* τ_2 *is nonoscillatory, too.*

PROOF. See for example [**12**]. □

As a corollary we get a proof of the observation above.

COROLLARY 4.20. *Let* $\tau u(n) = 0$ *have a nonoscillatory solution then any other solution is nonoscillatory.*

PROOF. $b_1(n) = b_2(n) = b(n)$ and therefore Sturm's comparison theorem yields the desired result. □

4.2. Nonoscillatory solutions and Infinite series

This section is based on the fact that minimal and maximal solutions of a Jacobi equation $\tau u(n) = 0$ are indicated by the divergence or convergence of the infinite series

$$\sum_{j=1}^{\infty} \frac{-1}{a(j)u(j)u(j+1)}. \tag{4.12}$$

This characterization of nonoscillatory solutions gives one the theory of convergence and divergence criteria for deriving conditions for a sequence to be a minimal or maximal solution. The most elementary criteria for convergence and divergence of infinite series rely on the following observation.

THEOREM 4.21. *Let* $\sum c(n)$ *be a given convergent series and let* $\sum d(n)$ *be a given divergent series then* $\sum s(n)$ *is*

(i) **convergent** *if* $s(n) \leq c(n)$ *and*
(ii) **divergent** *if* $s(n) \geq d(n)$.

Or, equivalently, if a series has a convergent majorant it is convergent and if it possesses a divergent minorant it diverges. The usefulness of such criteria relies on the choice of sequences $c(n)$ and $d(n)$. The convergent series should grow very fast and the divergent series should grow very slowly. The most important example for $c(n)$ and $d(n)$ are the *Abel series.* We refer to the Appendix 2 for definition and properties of Abel series, which are also called *logarithmic series.* The next theorem states the main theorem in this context.

THEOREM 4.22. *Let $\sum s(n)$ be a series of positive numbers, then*

$$s(n) \begin{array}{c} \leq \\ \geq \end{array} \frac{1}{n \cdot \log(n) \cdots \log_k(n)} \text{for} \begin{array}{c} \alpha > 1 \\ \alpha \leq 1 \end{array} \text{implies} \begin{array}{c} \text{convergence} \\ \text{divergence} \end{array} \tag{4.13}$$

of $\sum s(n)$.

In our case $s(n) = \big(a(n)u(n)u(n+1)\big)^{-1}$ and we are interested on monotone increasing solutions. These assumption allows us to derive explicit upper bounds for minimal and lower bounds for maximal solutions of Jacobi equations.

THEOREM 4.23. *Let u(n) be a positive solution of $\tau u(n) = 0$ that satisfies*

$$u(n) \leq \sqrt{\frac{1 \cdot n \cdots \log_k(n)}{a(n)}}, \tag{4.14}$$

then $u(n)$ is a minimal solution.

PROOF. The assumptions imply

$$1 \cdot n \cdots \log_k(n) \geq a(n)u(n)u(n+1) \tag{4.15}$$

and therefore

$$\sum_n^\infty \frac{1}{a(n)u(n)u(n+1)} \geq \sum_n^\infty \frac{1}{1 \cdot n \cdots \log_k(n)}. \tag{4.16}$$

And the last theorem implies

$$\sum_n^\infty \frac{1}{a(n)u(n)u(n+1)} = \infty, \tag{4.17}$$

i.e. the minimality of $u(n)$.

$\square$

The theorem of Abel and Dini, see Appendix 2, relates to a divergent series in a canonical way a convergent series, which in our case is

$$\sum_n^\infty \frac{1}{1 \cdot n \cdots \log_k(n) \log_{k+1}^{2\alpha}(n)}, \quad \alpha > \frac{1}{2}. \tag{4.18}$$

The last series is a natural choice for a convergent majorant and therefore we obtain the following lower bound for maximal solutions.

THEOREM 4.24. *Let $\tau u(n) = 0$ and $u(n)$ be a positive solution, that satisfies*

$$\hat{u}(n) \geq \sqrt{\frac{1 \cdot n \cdots \log_k(n)}{a(n)}} \log_{k+1}^\alpha(n), \quad \alpha > \frac{1}{2}, \tag{4.19}$$

then $u(n)$ is a maximal solution of $\tau u(n) = 0$.

PROOF. The assumption implies

$$a(n)u(n+1)u(n) \geq a(n)u(n)^2 \geq 1 \cdot n \cdots \log_k(n) \log_{k+1}^{2\alpha}(n), \tag{4.20}$$

and therefore

$$\sum_n^\infty \frac{1}{a(n)u(n)u(n+1)} \leq \sum_n^\infty \frac{1}{1 \cdot n \cdots \log_k(n) \log_{k+1}^{2\alpha}(n)}. \tag{4.21}$$

The assumption $\alpha > \frac{1}{2}$ implies the convergence of the right side of the last inequality and there fore $u(n)$ is a maximal solution. $\square$

Next we want to bring the following fact about convergent series to investigation of nonoscillatory solutions of Jacobi equations.

THEOREM 4.25. *Let $s(n)$ be a positive and monotone decreasing sequence with $\sum s(n) < \infty$. Then*

$$s(n) \to 0, \quad n \cdot s(n) \to 0, \quad \cdots \quad n \cdot \log(n) \cdots \log_k(n) s(n) \to 0, \tag{4.22}$$

if the convergence of $\sum s(n)$ is determined with the logarithmic scale.

In general the relation $ns(n) \to 0$ is only a necessary but not a sufficient condition for convergence, i.e. $ns(n) \to s \neq 0$ implies the divergence of $s(n)$.

PROOF. See [**14**] p.321, ex.141. $\square$

The application of this fact to Jacobi equations is the following theorem.

THEOREM 4.26. *Let $\tau u(n) = 0$ be a nonoscillatory Jacobi equation and $u(n)$ a maximal monotone increasing solution, then*

(i) $\frac{1}{a(n)u(n)u(n+1)} \to 0$,
(ii) $\frac{1 \cdot n}{a(n)u(n)u(n+1)} \to 0$,
(iii) $\frac{1 \cdot n \cdots \log_k(n)}{a(n)u(n)u(n+1)} \to 0$.

REMARK 4.27. *If $\frac{s(n)}{t(n)} \to 0$ one says that $t(n)$ tends faster to zero as $s(n)$ which in our case gives an explicit statement about the growth of maximal solutions, if we assume monotony of solutions.*

THEOREM 4.28. *A maximal and monotone increasing solution of a Jacobi equation $\tau u(n) = 0$ tends faster to zero as $\sqrt{\frac{1 \cdot n \cdot \log(n) \cdots \log_k(n)}{a(n)}}$*

PROOF. Reformulation of the last theorem. $\square$

4.3. Riccati equation

It is often useful to transform an object to reveal all its properties. In the case of Jacobi equations the following are of great usefulness. If $u(n) > 0$ has no zeros for $n \in \mathbb{N}$ we introduce

(i) $\phi_1(n) = \frac{u(n+1)}{u(n)}$,
(ii) $\phi_2(n) = a(n)\frac{u(n+1)}{u(n)}$,
(iii) $\phi_3(n) = \frac{b(n+1)u(n+1)}{a(n)u(n)}$,

called **Riccati transformations**, which lead to nonlinear difference equations, called **Riccati equations**

(i) $a(n)\phi_1(n) + a(n-1)\phi_1^{-1}(n-1) = b(n) - \lambda$,
(ii) $\phi_2(n) + a(n-1)^2\frac{1}{\phi_2(n-1)} = b(n) - \lambda$,
(iii) $q(n)\phi_3(n) + \frac{1}{\phi_3(n-1)} = 1 + \frac{\lambda}{b(n)}$, with $q(n) = \frac{a(n)^2}{b(n)b(n+1)}$

related to

$$a(n)u(n+1) + a(n-1)u(n-1) = b(n)u(n) - \lambda u(n). \tag{4.23}$$

REMARK 4.29. *The transformation* $\phi_2(n) = \frac{a(n+1)u(n+1)}{u(n)}$ *is perhaps the nearest analogue for difference equations to the classical Riccati transformation* $\phi(t) = \frac{x'(t)}{x(t)}$ *which transforms the self-adjoint differential equation* $(px')' + qx = 0$ *with* $p(t) \neq 0$ *into the Riccati equation*

$$\phi'(t) + \frac{\phi(t)^2}{p(t)} + q(t) = 0, \tag{4.24}$$

because of

$$\phi_2(n) + a(n-1)^2 \frac{1}{\phi_2(n-1)} = b(n) - \lambda, \tag{4.25}$$

$$\Delta\phi_2(n-1) + \frac{\phi_2(n-1)\phi_2(n)}{b(n)} - \phi_2(n) + \frac{a(n-1)^2}{b(n)} + \lambda = 0. \tag{4.26}$$

(4.27)

THEOREM 4.30. *The following conditions are equivalent:*

(i) τ *is oscillatory.*
(ii) $a(n)\phi_1(n) + \frac{a(n-1)}{\phi_1(n-1)} = b(n)$ *has a positive solution* $\phi_1(n)$, $\quad n \in \mathbb{N}$.
(iii) $\phi_2(n) + a(n-1)^2 \frac{1}{\phi_2(n-1)} = b(n)$ *has a positive solution* $\phi_2(n)$, $\quad n \in \mathbb{N}$.
(iv) $q(n)\phi_3(n) + \frac{1}{\phi_3(n-1)} = 1$, *has a positive solution* $\phi_3(n)$, $\quad n \in \mathbb{N}$.

PROOF. If $\tau u = 0$ is nonoscillatory then $u(n)u(n+1) > 0$ for all $n \in \mathbb{N}$ and the necessity follows immediately from the Riccati transformations. Now assume, that $\phi_1(n) > 0$ is a solution of $a(n)\phi_1(n) + \frac{a(n-1)}{\phi_1(n-1)} = b(n)$,then we may let $u(0) = 1$ and $u(n+1) = u(n)\phi_1(n)$ for all $n \in \mathbb{N}$. $u(n)$ is a positive solution and therefore τ is nonoscillatory. Similar arguments hold for the remaining equations. □

THEOREM 4.31. *If* $b(n)b(n+1) \leq (4-\varepsilon)a^2(n)$ *for some* $\varepsilon > 0$ *and* $n \in \mathbb{N}$ *then* τ *is oscillatory.*

PROOF. See [**1**], Thm.6.5.3. □

REMARK 4.32. *Consider the Jacobi equation*

$$\sqrt{n}u(n+1) + \sqrt{n}u(n-1) - (\sqrt{n+1} - \sqrt{n-1})u(n) = 0, \tag{4.28}$$

which is nonoscillatory since it has a solution $u_1(n) = \sqrt{n}$ *for* $n \in \mathbb{N}$. *But* $b(n)b(n+1) < 4$ *and* $\varepsilon(n) = 4 - b(n)b(n+1)$ *tends to zero as* $n \to \infty$ *and we have* $b(n)b(n+1) = 4 - \varepsilon(n)$, *but* τ *is nonoscillatory and therefore the inequality condition cannot be replaced by the weaker condition*

$$b(n)b(n+1) \leq (4 - \varepsilon(n))a^2(n), \tag{4.29}$$

where $\varepsilon(n) > 0$ *and* $\varepsilon(n) \to 0$ *as* $n \to \infty$.

THEOREM 4.33. *If* $b(n+1)b(n) \geq 4a^2(n)$ *for all* $n \in \mathbb{N}$ *then* τ *is nonoscillatory.*

PROOF. See [**1**], Thm.6.5.5. □

CHAPTER 5

Main results and applications

Before we can write down our main result, we need to fix some notation. Recall that τ is called oscillatory if one (and hence any) real-valued solution of $\tau u = 0$ has an infinite number of nodes, that is, points $n \in \mathbb{N}$, such that either

$$u(n) = 0 \quad \text{or} \quad a(n)u(n)u(n+1) > 0. \tag{5.1}$$

In the special case $a(n) < 0$, $n \in \mathbb{Z}$ a node of u is precisely a sign flip of u as one would expect. In the general case, however, one has to take the sign of $a(n)$ into account.

Recall that if $u_0(n) > 0$ solves

$$(\tau_0 u)(n) = a(n)u_0(n+1) + a(n-1)u_0(n-1) - b_0(n)u_0(n) = 0 \tag{5.2}$$

then

$$\hat{u}_0(n) = u_0(n)Q_0(n), \quad Q_0(n) = \sum_{j=0}^{n-1} \frac{-1}{a(j)u_0(j)u_0(j+1)}, \tag{5.3}$$

is a second, linearly independent positive solution. A positive solution is called minimal if

$$\lim_{n\to\infty} Q_0(n) = \infty. \tag{5.4}$$

Minimal solutions are unique up to a multiple. See [**20**], Section 2.3 for more information. With this notation our main result reads as follows:

THEOREM 5.1. *Suppose $a_0 < |a(n)| < A_0$. Let $u_0(n)$ be a monotone non-decreasing minimal positive solution of $\tau_0 u_0 = 0$ and abbreviate*

$$A(n) = \frac{2a(n-1)a(n+1)}{a(n-1)+a(n+1)}. \tag{5.5}$$

Then τ is nonoscillatory if

$$\liminf_{n\to\infty} -A(n)u_0^4(n)Q_0^2(n)(b(n)-b_0(n)) > -\frac{1}{4} \tag{5.6}$$

and oscillatory if

$$\limsup_{n\to\infty} -A(n)u_0^4(n)Q_0^2(n)(b(n)-b_0(n)) < -\frac{1}{4}. \tag{5.7}$$

The proof will be given in Section 5.1 below.

As a first application, let us show how this result can be used to answer our question posed in the introduction. We choose

$$a(n) = -1, \quad b_0(n) = 2. \tag{5.8}$$

Then we have

$$u_0(n) = 1 \quad \text{and} \quad \hat{u}_0(n) = n \tag{5.9}$$

and thus

$$\lim_{n\to\infty} \begin{matrix}\inf\\ \sup\end{matrix} \big(n^2(b(n)-2)\big) \begin{matrix}>\\ <\end{matrix} -\frac{1}{4} \text{ implies } \begin{matrix}\text{nonoscillation}\\ \text{oscillation}\end{matrix} \text{ of } \tau \text{ near } \infty, \tag{5.10}$$

which is the claimed generalization of Kneser's result. Clearly, the next question is what happens in the limiting case, where $\lim_{n\to\infty} n^2(b(n)-2) = -4^{-1}$? This can be answered by our result as well:

Recall the iterated logarithm

$$\ln_0(x) = x, \qquad \ln_k(x) = \ln_{k-1}(\ln(x)), \tag{5.11}$$

where $\ln_k(x)$ is defined for $x > \mathrm{e}_k$, with $\mathrm{e}_1 = 0$, $\mathrm{e}_k = \mathrm{e}^{\mathrm{e}_{k-1}}$.

Corollary 5.2. *Let*

$$a(n) = -1, \qquad b_k(n) = 2 - \frac{1}{4}\sum_{j=0}^{k-1} \frac{1}{\prod_{\ell=0}^{j} \log_\ell(n)^2}. \tag{5.12}$$

Then τ is nonoscillatory if

$$\liminf_{n\to\infty} \Big(\prod_{j=0}^{k} \log_j(n)\Big)^2 (b(n) - b_k(n)) > -\frac{1}{4} \tag{5.13}$$

and oscillatory if

$$\limsup_{n\to\infty} \Big(\prod_{j=0}^{k} \ln_j(n)\Big)^2 (b(n) - b_k(n)) < -\frac{1}{4}. \tag{5.14}$$

Proof. To show how this follows from our result we consider

$$u_k(n) = \sqrt{\prod_{j=0}^{k} \log_j(n)}, \tag{5.15}$$

which is a solution of $\tilde{\tau}_k$ associated with

$$a(n), \qquad \tilde{b}_k(n) = \frac{u_k(n+1) + u_k(n-1)}{u_k(n)}. \tag{5.16}$$

To prove the claim it suffices to show

$$\begin{aligned} \tilde{b}_k(n) &= b_k(n) + O(n^{-3}) \\ Q_k(n) &= \log_k(n) + O(1) \end{aligned} \tag{5.17}$$

since the differences will not contribute to the limits from above.

To establish (5.17) we first recall the following formulas for the first and second derivative of $\ln_k$:

$$\begin{aligned} \ln_k'(x) &= \prod_{j=0}^{k-1} \frac{1}{\log_j(x)}, \\ \ln_k''(x) &= -\log_k'(x) \sum_{j=1}^{k} \ln_j'(x), \qquad x > \mathrm{e}_k. \end{aligned} \tag{5.18}$$

Now we can show (5.17). First of all we have

$$Q_k(n) = \int^n \frac{dx}{u_k(x)^2} + O(1) = \log_k(n) + O(1). \tag{5.19}$$

The second claim is a bit harder. We begin with

$$\begin{aligned}
\frac{u_k(n\pm 1)}{u_k(n)} &= \left(\prod_{j=0}^{k} \frac{\log_j(n\pm 1)}{\log_j(n)}\right)^{1/2} = \prod_{j=0}^{k}\left(\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!}\frac{\log_j^{(\ell)}(n)}{\log_j(n)}\right)^{1/2} \\
&= \prod_{j=0}^{k}\left(1 \pm \frac{\ln_j'(n)}{\ln_j(n)} + \frac{1}{2}\frac{\ln_j''(n)}{\ln_j(n)} + O(n^{-3})\right)^{1/2} \\
&= \prod_{j=0}^{k}\left(1 \pm \frac{1}{2}\log_{j+1}'(n) + \frac{1}{4}\left(\frac{\log_j''(n)}{\ln_j(n)} - \frac{\log_{j+1}'(n)^2}{2}\right) + O(n^{-3})\right) \\
&= 1 \pm \frac{1}{2}\sum_{j=0}^{k}\log_{j+1}'(n) + \frac{1}{4}\sum_{j=0}^{k}\left(\frac{\log_j''(n)}{\ln_j(n)} - \frac{\log_{j+1}'(n)^2}{2}\right) + \\
&\quad + \frac{1}{4}\sum_{j=0}^{k}\log_{j+1}'(n)\sum_{\ell=0}^{j-1}\log_{\ell+1}'(n) + O(n^{-3}). \qquad (5.20)
\end{aligned}$$

Now combining both formulas we obtain the desired result

$$\begin{aligned}
\tilde{b}_k(n) &= 2 - \frac{1}{2}\sum_{j=0}^{k}\left(\log_{j+1}'(n)\sum_{\ell=1}^{j}\ln_\ell'(n) + \frac{\log_{j+1}'(n)^2}{2} - \sum_{\ell=0}^{j-1}\log_{\ell+1}'(n)\right) + O(n^{-3}) \\
&= b_k(n) + O(n^{-3}). \qquad (5.21)
\end{aligned}$$

□

Another interesting example is the case

$$b_0(n) = -a(n) - a(n-1). \qquad (5.22)$$

Again we can take $u_0(n) = 1$ to obtain

Corollary 5.3. *Let $a_0 \le |a(n)| \le A_0$ and abbreviate*

$$A(n) = \frac{2a(n-1)a(n+1)}{a(n-1)+a(n+1)}, \qquad Q_0(n) = \sum_{j=0}^{n-1}\frac{-1}{a(j)}. \qquad (5.23)$$

Then τ is nonoscillatory if

$$\liminf_{n\to\infty} -A(n)Q_0(n)^2(b(n) + a(n-1) + a(n)) > -\frac{1}{4} \qquad (5.24)$$

and oscillatory if

$$\limsup_{n\to\infty} -A(n)Q_0(n)^2(b(n) + a(n-1) + a(n)) < -\frac{1}{4}. \qquad (5.25)$$

Of course one could take two arbitrary sequence $a(n) < 0$ and $u_0(n) > 0$ such that u_0 is non-decreasing and (5.4) is satisfied, compute $b_0(n) = -(a(n)u_0(n+1)+a(n-1)u_0(n-1))/u_0(n)$, and apply Theorem 5.1 to get a new (non)oscillation criterion.

5.1. Proof of the main theorem

We assume

$$a_0 \le |a(n)| \le A_0, \qquad b_0(n) \tag{5.26}$$

are given and that u_0 is a minimal positive non-decreasing solution of $\tau_0 u_0 = 0$ as in the previous section. Note that the corresponding second positive solution $\hat{u}_0(n)$ is increasing.

First we collect some basic facts which will be needed later on.

LEMMA 5.4. *Let u_0 be a minimal positive non-decreasing solution, then we have*

$$\lim_{n\to\infty} \frac{u_0(n+1)}{u_0(n)} = \lim_{n\to\infty} \frac{u_0(n)}{u_0(n-1)} = 1 \tag{5.27}$$

and

$$u_0(n)\hat{u}_0(n) = u_0^2(n) Q_0(n) \ge \frac{n}{A_0}. \tag{5.28}$$

PROOF. Monotonicity of u_0 implies

$$\frac{1}{u_0(j+1)^2} \le \frac{1}{u_0(j)u_0(j+1)} \le \frac{1}{u_0(j)^2} \tag{5.29}$$

Summing the last expression from 0 to $n-1$ and subtracting the right side yields

$$0 \le \sum_{j=0}^{n-1} \frac{1}{u_0(j)u_0(j+1)} \left(\frac{u_0(j+1)}{u_0(j)} - 1 \right) \le \frac{1}{u_0(0)^2} - \frac{1}{u_0(n)^2} \le \frac{1}{u_0(0)^2} \tag{5.30}$$

Since $u(n)$ is minimal,

$$\sum_{j=0}^{n-1} \frac{1}{u_0(j)u_0(j+1)} \ge a_0 Q_0(n) \to \infty \tag{5.31}$$

implies the first result.

For the second claim we use

$$Q_0(n) = \sum_{j=0}^{n-1} \frac{-1}{a(j)u_0(j)u_0(j+1)} \ge \frac{1}{A_0} \sum_{j=0}^{n-1} \frac{1}{u_0(j+1)^2} \ge \frac{n}{A_0 u_0(n)^2} \tag{5.32}$$

finishing the proof. □

Our next goal is to find a suitable comparison equation. We do this by trying the ansatz

$$u_1(n) = u_0(n) Q_0(n)^{\alpha}. \tag{5.33}$$

Then $u_1(n)$ satisfies

$$\tau_1 u(n) = a(n)u_1(n+1) + a(n-1)u_1(n-1) + b_1(n)u_1(n) = 0 \tag{5.34}$$

with $b_1(n)$ given by

$$\begin{aligned} b_1(n) \quad = \quad & -\frac{a(n)u_0(n+1)}{u_0(n)} \left(1 - \frac{1}{a(n)u_0(n+1)^2 Q_0(n)} \right)^{\alpha} \\ & - \frac{a(n-1)u_0(n-1)}{u_0(n)} \left(1 + \frac{1}{a(n-1)u_0(n-1)^2 Q_0(n)} \right)^{\alpha}. \end{aligned} \tag{5.35}$$

In order to get an oscillating comparison equation we need to admit $\alpha \in \mathbb{C}$. However, this will also render $b_1(n)$ complex and hence it will be of no use for us.

To overcome this problem we look at the asymptotic behavior of $b_1(n)$ for $n \to \infty$, which is given by

$$b_1(n) = b_0(n) + \mu U(n) + O(\frac{1}{u_0^6(n) Q_0^3(n)}), \mu = \alpha(\alpha - 1), \tag{5.36}$$

where

$$U(n) = \frac{1}{2u_0^4(n) Q_0^2(n)} \left(\frac{-u_0(n)}{a(n+1)u_0(n+1)} + \frac{-u_0(n)}{a(n-1)u_0(n-1)} \right). \tag{5.37}$$

If $\alpha \in \mathbb{R}$ we can choose $b_1(n)$ directly as comparison potential to obtain that τ is nonoscillatory if

$$\liminf_{n\to\infty} \frac{b(n) - b_0(n)}{b_1(n) - b_0(n)} > \mu. \tag{5.38}$$

Using the optimal value $\alpha = \frac{1}{2}$ plus the expansion from above we end up with

$$\liminf_{n\to\infty} \frac{b(n) - b_0(n)}{U(n)} > -\frac{1}{4}. \tag{5.39}$$

This settles the first part of our theorem. Now we come to the harder one. As already noticed, in order to get an oscillating comparison equation we need to choose complex values for α. Our strategy is to choose $\alpha = \frac{1}{2} + \mathrm{i}\varepsilon$ such that at least $\mu = -\frac{1}{4} - \varepsilon^2$ remains real and take $\tilde{b}_1(n) = b_0(n) + \mu U(n)$ as comparison equation. Of course we don't know the solutions of this equation, but our hope is that they asymptotically are given by the real/imaginary parts of

$$u_1(n) = u_0(n)\sqrt{Q_0(n)} \left(\cos(\varepsilon \ln Q_0(n)) + \mathrm{i} \sin(\varepsilon \ln Q_0(n)) \right). \tag{5.40}$$

Hence if we can show that there are solutions $\tilde{u}_1$ of $\tilde{\tau}_1 \tilde{u}_1 = 0$ satisfying

$$\tilde{u}_1(n) = u_1(n)(1 + O(\frac{1}{n})) \tag{5.41}$$

we are done.

To show this we begin with

$$\tau_1 \tilde{u}_1(n) = \Delta(n) \tilde{u}_1(n), \qquad \Delta(n) = b_1(n) - \tilde{b}_1(n), \tag{5.42}$$

and rewrite this equation as (compare [**20**], Section 1.1)

$$\tilde{u}_1(n) = u_1(n) - \sum_{j=n+1}^{\infty} u_1(n) u_1(j) (Q_1(n) - Q_1(j)) \Delta(n) \tilde{u}_1(n). \tag{5.43}$$

Moreover, setting

$$\tilde{u}_1(n) = u_1(n) v(n) \tag{5.44}$$

we obtain

$$v(n) = 1 - \sum_{j=n+1}^{\infty} u_1(n)^2 (Q_1(n) - Q_1(j)) \Delta(n) v(n). \tag{5.45}$$

To show existence of a solution $v(n) = 1 + o(1)$ it remains to verify the assumptions of [**20**], Lemma 7.8. To do this we need to estimate the kernel of the above sum equation. Using

$$\begin{aligned} |Q_1(n) - Q_1(j)| &\leq \sum_{k=n}^{j} \frac{1}{a(k)u_0(k)u_0(k+1)\sqrt{Q_0(k)Q_0(k+1)}} \\ &\leq \frac{1}{a_0} \sum_{k=n}^{j} \frac{1}{u_0(k)^2 Q_0(k)} \end{aligned} \tag{5.46}$$

and

$$|\Delta(n)| \leq \frac{const}{u_0(j)^6 Q_0(j)^3} \tag{5.47}$$

we obtain by Lemma 5.4

$$\begin{aligned} |u_1(n)^2 (Q_1(n) - Q_1(j))\Delta(n)| &\leq \frac{const}{u_0(j)^4 Q_0(j)^2} \sum_{k=n}^{j} \frac{1}{u_0(k)^2 Q_0(k)} \\ &\leq const \frac{\ln(j)}{j^2}. \end{aligned} \tag{5.48}$$

Thus we can apply [**20**], Lemma 7.8 to conclude existence of a solution of type (5.41) which finishes the proof.

CHAPTER 6

Hardy's inequality

6.1. Hardy's original inequality

We follow the classic treatment [**7**].

THEOREM 6.1. *Let $a(n) > 0$ and $A(n) = a(1) + a(2) + \cdots + a(n)$, then for $p > 1$*

$$\sum_{n=1}^{\infty} \left(\frac{A(n)}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a(n)^p. \tag{6.1}$$

PROOF. Without loss of generality, we assume $a(n)$ decreases and that the continuous analog

THEOREM 6.2. *If $f(x) \geq 0$, and $F(x) = \int_0^x f(t)dt$, then*

$$\int_0^{\infty} \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f^p(x)dx, \tag{6.2}$$

unless $f \equiv 0$.

has been proved. The restriction $a(n)$ decreases causes no loss of generality, because the following holds:

LEMMA 6.3. *If the $a(n)$ are given except in arrangement, and $\phi(t)$ is a positive increasing function, then*

$$\sum_{n=1}^{\infty} \phi\left(\frac{A(n)}{n}\right) \tag{6.3}$$

is greatest when the $a(n)$ are arranged in decreasing order.

PROOF. If $p > q$ and $a(p) > a(q)$, the effect of exchanging $a(p)$ and $a(q)$ is to leave $A(n)$ unchanged when $n < q$ or $n \geq p$, and to increase $A(n)$ when $q \leq n < p$. □

We define $f(x)$ by

$$f(x) = a(n), \quad n - 1 \leq x < n, \tag{6.4}$$

then

$$\sum_{n=1}^{\infty} a(n)^p = \int_0^{\infty} f^p(x)dx. \tag{6.5}$$

If $n < x < n + 1$, then

$$\frac{F(x)}{x} = \frac{a(1) + \cdots + a(n) + (x - n)a(n+1)}{x} = \frac{A(n) - na(n+1) + xa(n+1)}{x} \tag{6.6}$$

and

$$A(n) - na(n+1) \geq 0, \tag{6.7}$$

so that $F(x)/x$ decreases from $A(n)/n$ to $A(n+1)/(n+1)$ when x increases from n to $n+1$. Hence

$$\frac{F(x)}{x} \geq \frac{A(n+1)}{n+1} \tag{6.8}$$

and so

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \geq \sum_{n=1}^\infty \left(\frac{A(n)}{n}\right)^p. \tag{6.9}$$

From

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx \tag{6.10}$$

we obtain

$$\sum_{n=1}^\infty \left(\frac{A(n)}{n}\right)^p \leq \int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx = \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty a(n)^p. \tag{6.11}$$

$\square$

REMARK 6.4. *Hardy's inequality has an important corollary, the inequality of Carleman: If we write $a(n)$ for $a(n)^p$, we obtain*

$$\sum \left(\frac{a(1)^{1/p} + \cdots + a(n)^{1/p}}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum a(n). \tag{6.12}$$

If we make $p \to \infty$ and use the inequality of the geometric and arithmetic mean, we obtain

THEOREM 6.5.

$$\sum (a(1)\cdots a(n))^{1/n} \leq e \sum a(n). \tag{6.13}$$

The last inequality is Carleman's inequality.

REMARK 6.6. *In the words of functional analysis Hardy's inequality states as*

(i)

$$(Tf)(x) = \frac{1}{x}\int_0^x f(y)dy \tag{6.14}$$

is a map in $L^p(0,\infty)$ of norm $\frac{p}{p-1}$ for $p > 1$.

(ii)

$$T : a(n) \mapsto b(n) = \frac{1}{n}\sum_{j=0}^n a(j) \tag{6.15}$$

is a map of $\ell^p(\mathbb{N})$ of norm $\frac{p}{p-1}$ for $p > 1$.

6.2. Generalization of Hardy's inequality

F. Gesztesy and M. Ünal proved a generalization of Hardy's inequality for the case $p = 2$, [**5**].

THEOREM 6.7. *Let $\psi_0(x)$ be a positive solution of $(p_0\psi')' - q_0\psi = 0$.Let $\int^\infty \frac{dt}{p_0(t)\psi_0(t)^2} = \infty$ and $\int_a \frac{dt}{p_0(t)\psi_0(t)^2} < \infty$ and $\psi_0(x) > 0$ on (a,∞). Then for all $0 \neq \phi \in C_0^\infty\big((a,\infty)\big)$,*

$$\int_a^\infty p_0(x)|\phi'(x)|^2 dx$$
$$> \int_a^\infty [\frac{1}{4p_0(x)\psi_0(x)^4\big(\int_a^x \frac{dt}{p_0(t)\psi_0(t)^2}\big)^2} - q_0(x)]|\phi(x)|^2 dx. \tag{6.16}$$

The constant $1/4$ is sharp.

This is a general statement about Sturm-Liouville equation, but to be more concrete, one needs Sturm-Liouville equations with known solutions. The simplest example provides $p_0 = 1$ and $q_0 = 0$ then $\psi_0(x) = 1$ is a positive solution of $\psi" = 0$ and $\int^\infty \frac{dt}{\psi_0^2(t)} = \infty$ and therefore all assumptions of the last theorem are fulfilled and we get Hardy's inequality

$$\int_a^\infty \phi'^2(x)dx > \frac{1}{4}\int_a^\infty \frac{\phi^2(x)}{x^2}dx. \tag{6.17}$$

The simplest nonoscillatory equation gives Hardy's inequality and therefore one gets generalizations of Hardy's inequality if one uses other nonoscillatory equations with known solutions. M. Ünal states in his thesis, [**22**], a whole scale of nonoscillatory equations with known solutions. We only state the result and refer to [**22**] p.22 Lem.3.2 for details.

LEMMA 6.8.

$$\psi_n(x) = \big(\prod_{k=0}^{n-1} \log_k(x)\big)^{1/2}(x), \tag{6.18}$$

is a solution of the equation

$$-\psi_n''(x) + q_n(x)\psi_n(x) = 0, \quad x \neq 0, \tag{6.19}$$

where

$$q_n(x) = -\Big[\sum_{k=1}^{n}\big(\prod_{j=0}^{k-1} \log_j(x)\big)^{-2}\Big]/4, \tag{6.20}$$

or more precisely,

$$q_1(x) = -\frac{1}{4x^2}\ , \quad q_2(x) = -\frac{1}{4x^2} - \frac{1}{4x^2(\log(|x|))^2}\ , \ldots,\ etc. \tag{6.21}$$

The last lemma combined with the last theorem gives following Hardy-type inequality

THEOREM 6.9. *Let $0 \neq \phi \in C_0^\infty\big((a,\infty)\big)$ and $\Phi(x) = \int_a^x \phi(t)dt$,*

$$\int_a^\infty \phi^2(x) > \frac{k}{4}\int_a^\infty \frac{\Phi^2(t)}{\big(\prod_{j=0}^{k-1} \log_k(x)\big)^{-2}}, \tag{6.22}$$

for $a > e_k$.

We obtained the discrete Hardy inequality from the continuous version via estimates relying on the behavior of the solutions of $\psi"(x) = 0$ therefore the last theorem is the key ingredient in the proof of the next theorem.

THEOREM 6.10. *Let $a(n)$ be a positive sequence and $A(n) = \sum_{j=0}^{n} a(j)$ then*

$$\sum_{e_k+1}^{\infty} a(n)^2 > \frac{k}{4} \sum_{e_k+1}^{\infty} \frac{A(n)^2}{n \cdot \log(n \cdots \log_{k-1}(n))}. \tag{6.23}$$

PROOF. W.l.o.g we assume $a(n)$ to be monotone decreasing and the argument is based on the lemma used in the proof of Hardy's inequality in the last chapter. Let $\phi(x) = a(n) \quad n \leq x < n+1$, then

$$\sum_{1}^{\infty} a(n)^2 = \int_0^{\infty} \phi(t)^2 dt. \tag{6.24}$$

If $n < x < n+1$ then

$$\frac{\Phi(x)}{x \cdot \log(x) \cdots \log_{k-1}(x)} = \frac{A(n) - na(n+1) + xa(n)}{x \cdot \log(x) \cdots \log_{k-1}(x)}, \tag{6.25}$$

but $a(n)$ is monotone decreasing and therefore

$$\frac{\Phi(x)}{x \cdot \log(x) \cdots \log_{k-1}(x)} \geq \frac{\Phi(n+1)}{(n+1) \cdot \log(n+1) \cdots \log_{k-1}(n+1)}. \tag{6.26}$$

The last inequality yields

$$\int_{e_k}^{\infty} \left(\frac{\Phi(x)}{x \cdot \log(x) \cdots \log_{k-1}(x)}\right)^2 \geq \sum_{e_k+1}^{\infty} \left(\frac{A(n)}{n \cdot \log(n) \cdots \log_{k-1}(n)}\right)^2, \tag{6.27}$$

which implies

$$\begin{aligned} \sum_{n=e_k+1}^{\infty} a(n)^2 = \int_{e_k}^{\infty} \phi(t)^2 dt \quad &\geq \quad \int_{e_k}^{\infty} \left(\frac{\Phi(x)}{x \cdot \log(x) \cdots \log_{k-1}(x)}\right)^2 \\ &\geq \quad \sum_{e_k+1}^{\infty} \left(\frac{A(n)}{n \cdot \log(n) \cdots \log_{k-1}(n)}\right)^2, \end{aligned} \tag{6.28}$$

our desired result. $\square$

CHAPTER 7

Appendices

7.1. Appendix 1

This section contains some material about differences and difference equations and their relations to differentials and differential equations. The exposition will be more informal as the former chapters.

7.1.1. Differences and Difference Equations. What is the first difference of a function? Fix a number $h > 0$ (called the **step size**). The first difference of the given function φ with step size h is the function whose value at the point t is $\varphi(t+h) - \varphi(t)$. The first difference is denoted $\Delta\varphi$. The second difference $\Delta^2\varphi$ is defined as $\Delta(\Delta\varphi)$. This gives

$$\Delta(\Delta\varphi) = \varphi(t+2h) - 2\varphi(t+h) + \varphi(t). \tag{7.1}$$

The n^{th} difference is defined similarly: $\Delta^n\varphi = \Delta(\Delta^{n-1}\varphi)$. As $h \to 0$ the expression $\frac{\Delta\varphi}{\Delta t}$ becomes the derivative of φ at the point t. And analog is the $n-th$ derivative of φ at the point t the limit of the difference quotient of the $n-th$ order as $h \to 0$.

A first order difference equation is an equation of the form $\frac{\Delta\varphi}{\Delta t} = f(t;\varphi(t))$. From such an equation knowing only the number $\varphi(t_0)$, it is possible to find $\varphi(t_0+h)$, and from the latter $\varphi(t_0+2h)$, etc. As $h \to 0$ a difference equation becomes a differential equation. It is therefore not surprising that the solution of a first order differential equation is also determined by the value of one single number at the initial instant. A **second order difference equation** has the form

$$\frac{\Delta^2\varphi}{(\Delta t)^2} = \frac{\varphi(t+2h) - 2\varphi(t+h) + \varphi(t)}{h^2} = F\big(t;\varphi,\frac{\Delta\varphi}{\Delta t}\big). \tag{7.2}$$

Knowing the value of φ at two instants separated by a time interval of length h, we can find the value of φ after another interval h from this equation. Thus all the values $\varphi(t+kh)$ are determined by the first two of them. As $h \to 0$ the second order difference equation becomes a second order differential equation. And so the differential equation is also determined by giving two numbers at the initial instant. And last, a n^{th} **order difference equation** is an expression of the form $\frac{\Delta^n\varphi}{(\Delta t)^n} = F\big(t;\varphi,...,\frac{\Delta^{n-1}\varphi}{(\Delta t)^{n-1}}\big)$ and n is determined by giving n numbers. It also becomes a differential equation for $h \to 0$.

In the last section we expressed the derivative of a function in terms of differences with arbitrary small step size. It is also possible to express derivatives in terms of differences of a fixed value of h. If $\varphi(t)$ is an analytic function of t, then by Taylor's theorem

$$\varphi(t+h) = \varphi(t) + h\varphi'(t) + \frac{h^2}{2!}\varphi''(t) + \cdots. \tag{7.3}$$

Let D denote differentiation; thus $D\varphi(t) = \varphi'(t)$. Then Taylor's theorem may be written as

$$\varphi(t+h) = [1 + hD + \frac{h^2}{2!}D^2 + \cdots]\varphi(t), \tag{7.4}$$

but $\varphi(t+h) = \varphi(t+h) - \varphi(t) + \varphi(t) = (1+\Delta)\varphi(t)$, which yields the formula

$$(1+\Delta)\varphi(t) = e^{hD}\varphi(t). \tag{7.5}$$

Suppressing $\varphi(t)$, we get the operational relation

$$e^{hD} = 1 + \Delta. \tag{7.6}$$

Or the following equivalent expression

$$hD = \log(1+\Delta) = \Delta - \frac{\Delta^2}{2} + - \cdots ; \tag{7.7}$$

applying this to $\varphi(t)$, we obtain the desired formula for the derivative of $\varphi(t)$ in terms of differences, namely,

$$\varphi'(t) = \frac{1}{h}[\Delta\varphi(t) - \frac{1}{2}\Delta^2 + - \cdots]. \tag{7.8}$$

Note that the relation $e^{hD} = 1 + \Delta$ gives directly

$$\Delta\varphi = (e^{hD} - 1)\varphi = (hD + \frac{h^2}{2}D^2 + \cdots)\varphi(t). \tag{7.9}$$

7.1.2. Inverse of a Difference Operator. Suppose we are given a function $\psi(t)$ such that

$$\varphi(t+1) - \varphi(t) = \psi(t) \tag{7.10}$$

for all values of t. (Hereafter, unless otherwise stated, we assume without loss of generality differences of length one.) How can we find $\psi(t)$? Formally, we may write

$$\varphi(t) = \Delta^{-1}\psi(t), \tag{7.11}$$

but this is meaningless until Δ^{-1} has been interpreted. To do so, let us return to the definition of difference. The definition gives

$$\begin{aligned} \varphi(t) - \varphi(t-1) &= \psi(t-1) \\ \varphi(t-2) - \varphi(t-1) &= \psi(t-2) \\ \varphi(1) - \varphi(0) &= \psi(0). \end{aligned}$$

Adding these equations together, we find that

$$\varphi(t) - \varphi(0) = \sum_{k=0}^{t-1}\psi(k) = \Delta^{-1}\psi(t) - \varphi(0). \tag{7.12}$$

Thus the inverse of differencing is summing. This corresponds to the fact that the inverse of differentiating is integrating. Notice that in summation, as in integration, the result is uniquely defined except for the additive constant $\varphi(0)$.

7.1.3. Difference Calculus. In analogy to the calculus of differentiation you can develop a calculus of differences.

(i) $\Delta(\varphi_1 + \varphi_2) = \Delta\varphi_1 + \Delta\varphi_2$
(ii) $\Delta(c\varphi) = c\Delta\varphi$

The last to properties are obtained by simple computation, but the consequences are of fundamental importance. If we consider Δ as an operator acting on a set of functions. The essence of the last to relations is the linearity of the operation of taking differences.
The difference of a product and of a quotient of two functions φ_1, φ_2 is

(i) $\Delta(\varphi_1\varphi_2)(t) = \Delta\varphi_1(t)\varphi_2(t) + \varphi_1(t+1)\Delta\varphi_2(t) = \varphi_1(t+1)\Delta\varphi_2(t) + \Delta\varphi_1(t)\varphi_2(t)$
(ii) $\Delta(\frac{\varphi_1}{\varphi_2}) = \frac{\Delta\varphi_1(t)\varphi_2(t) - \varphi_1(t+1)\Delta\varphi_2(t)}{\varphi_2(t)\varphi_2(t+1)}$

REMARK 7.1. *The results can be generalized to the product or quotient of n functions.*

What functions are invariant under taking differences, i.e what is the analogue of the exponential e^t? Two classes of functions are invariant under differences:

(i) The periodic functions with primitive period 1 (or in general with primitive period h) and
(ii) $\varphi(t) = 2^t$, since $\varphi(t+1) - \varphi(t) = \varphi(t)$ gives $\varphi(t+1) = 2\varphi(t)$ and the initial condition $\varphi(0) = 1$.

REMARK 7.2. *The last result establishes another connection between difference and differential equations, because e^x is invariant under differentiation and 2^n is invariant under taking forward differences, but $2^n = \lfloor e^n \rfloor$.*

7.2. Appendix 2

7.2.1. The logarithmic scale. The iterated logarithm occurs in theorems about oscillation criteria for differential and difference equations or about convergence and divergence criteria for series. The reason for its appearance in convergence criteria is the subject of this section. Our presentation is mainly based on Knopp, [**14**]. The following theorems are the key of an understanding, why the logarithmic scale is of importance in the theory of series and gives a justification of the statement *"The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is the simplest divergent series."*.

THEOREM 7.3. *(N.H. Abel, U. Dini)*

Let (d_n) be a real, positive sequence with divergent series $\sum d_n$ and $D_n = \sum_{j=0}^{n} d_j$ then

$$\sum_{j=1}^{\infty} \frac{d_j}{D_j^{\alpha}} \quad \begin{array}{ll} \alpha > 1 & \text{convergent} \\ \alpha \leq 1 & \text{divergent.} \end{array} \tag{7.13}$$

We only present a proof of the divergence part to give a feeling for proofs of such theorems and refer for the proof of the convergence to [**14**] p. 301.

PROOF. First we observe that the divergence of the series for $\alpha = 1$ implies the divergence for $\alpha < 1$, since D_n is monotone increasing and $D_n^\alpha \le D_n$ for $\alpha < 1$ yields

$$\sum_{j=1}^{\infty} \frac{d_j}{D_j} \le \sum_{j=1}^{\infty} \frac{d_j}{D_j^\alpha}, \tag{7.14}$$

but $\sum_{j=1}^{\infty} \frac{d_j}{D_j}$ is a divergent minorant for $\sum_{j=1}^{\infty} \frac{d_j}{D_j^\alpha}$ and therefore $\sum_{j=1}^{\infty} \frac{d_j}{D_j^\alpha}$ is divergent for $\alpha < 1$. Now we proof the divergence for $\alpha = 1$.

$$\frac{d_{n+1}}{D_{n+1}} + \cdots + \frac{d_{n+k}}{D_{n+k}} \ge \frac{d_{n+1} + \cdots + d_{n+k}}{D_n} = 1 - \frac{D_n}{D_{n+k}}. \tag{7.15}$$

According to the assumption $D_n \to \infty$ implies that we can always find a k_n to a given n with

$$\frac{D_n}{D_{n+k_n}} < \frac{1}{2} \quad \text{for} \quad k \ge k_n. \tag{7.16}$$

The last inequality implies

$$\frac{d_{n+1}}{D_{n+1}} + \cdots + \frac{d_{n+k}}{D_{n+k}} \ge \frac{1}{2} \quad for \quad k \ge k_n, \tag{7.17}$$

and therefore $\big(\sum_{j=1}^{n} \frac{d_j}{D_j}\big)$ is not a Cauchy sequence and therefore divergent. □

The simplest choice for d_n is to set $d_n \equiv 1$ then $D_n = n$ and the last theorem gives us a complete characterization of the convergence and divergence behavior of $\sum_{j=1}^{\infty} \frac{1}{n^\alpha}$ and a justification of the statement that $\sum_{n=1}^{\infty} \frac{1}{n}$ is the simplest divergent series.

REMARK 7.4. *A replacement of D_n' for D_n, where D_n' is asymptotically equivalent to D_n does not cause any changes in the convergence behavior as stated in the last theorem.*

The theorem of Abel and Dini taught us that divergence of $\sum d_n$ implies that of $\sum \frac{d_n}{D_n}$ but what is the relation between these two series? The next theorem gives an answer under the condition $\frac{d_n}{D_n} \to 0$, that is fulfilled if the d_n are bounded from above. A condition, that most of the studied sequences satisfy.

THEOREM 7.5. *Let d_n and D_n be as in the theorem of Abel and Dini and assume $\frac{d_n}{D_n} \to 0$ then*

$$\frac{d_1}{D_1} + \cdots + \frac{d_n}{D_n} \approx \log D_n. \tag{7.18}$$

The proof is a nonelementary and relies on some not so commonly known facts about limits.For a proof we therefore refer the reader to [**14**]. As a corollary we obtain the asymptotic behavior of $\sum \frac{1}{n}$.

COROLLARY 7.6.

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \log n. \tag{7.19}$$

This result and the remark to Abel-Dini's theorem enables us to choose $d_n = \frac{1}{n}$ which yields:

$$\sum_{j=1}^{\infty} \frac{1}{n \log^{\alpha} n} \quad \begin{array}{ll} \alpha > 1 & \text{convergent} \\ \alpha \leq 1 & \text{divergent.} \end{array} \tag{7.20}$$

Iteration of the last two steps gives a scale of series, where each series is proceeded by one, that is more slowly convergent or divergent, resp. to α. The series are built out of $\log x$ and its iterations and therefore is called *logarithmic series*. We introduce $\log_k x = \log(\log_{k-1}(x))$ for $k > 1$ and $e_k(x) = e_{k-1}(e^x)$ to give the following results a simpler form.

THEOREM 7.7.

$$\sum_{n=e_k+1}^{n} \frac{1}{n \log(n) \cdots \log_k(n)} \approx \log_{k+1}(n). \tag{7.21}$$

The next result is the reason for the importance of the logarithmic scale.

THEOREM 7.8.

$$\sum_{n=e_k+1}^{\infty} \frac{1}{n \log(n) \cdots \log_k^{\alpha}(n))} \quad \begin{array}{ll} \alpha > 1 & \textit{convergent} \\ \alpha \leq 1 & \textit{divergent.} \end{array} \tag{7.22}$$

A nice application of the last theorem is a complete characterization of the behavior of

$$\sum_{n=e_k+1}^{\infty} \frac{1}{n^{\alpha_0} \log^{\alpha_1}(n) \cdots \log_k^{\alpha_k}(n)}, \tag{7.23}$$

often called *Abel Series.*

THEOREM 7.9. *Let $\alpha_0, \alpha_1, \cdots, \alpha_k$ be arbitrary real numbers then*

$$\sum_{n=e_k+1}^{\infty} \frac{1}{n^{\alpha_0} \log^{\alpha_1}(n) \cdots \log_k^{\alpha_k}(n)} \tag{7.24}$$

is convergent if the first from 1 different exponent is greater than 1 and is divergent if all α_j are less or equal 1.

REMARK 7.10. *The last theorem gives the convergence of*

$$\sum_{n=e_k(+1)}^{\infty} \frac{1}{n^2 \log^2(n) \cdots \log^2(n)_k(n)}, \tag{7.25}$$

because $\alpha_k = 2$ for $k = 0, 1, \cdots, p$. The last series is of great importance in our treatment of oscillation criteria for Jacobi equations.

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